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Applying the Modified Generalized Laguerre Functions for Solving Steady Flow of a Third Grade Fluid in a Porous Half Space

Kourosh Parand and Fattaneh Bayat Babolghani

Department of Computer Sciences, Shahid Beheshti University, Tehran, Iran

Abstract: In this paper we provide a collocation method for the problem of steady flow of a third grade fluid in a porous half space. This problem is a non-linear, two-point boundary value problem (BVP) on semi-infinite interval. This approach is based on a modified generalized Laguerre which is an orthogonal function. We also present the comparison of this work with solution of other methods; moreover, in the graph of the $||\mathrm{Re}s||^2$, we show that the present solution is more accurate and faster convergence in this problem.

AMS Subject Classifications: 34B15 · 34B40
Key words: Steady flow · Third grade fluid · Orthogonal function · Modified generalized Laguerre functions · Spectral methods · Semi-infinite

INTRODUCTION

The flow of non-Newtonian fluids has several technical applications, especially in the paper and textile industries. Out of many models which have been used to describe the non-Newtonian behavior exhibited by certain fluids. The fluids of the differential type have received special attention. Fluids of the second and the third grade have been studied in various types of flow situations which form a subclass of the fluids of the differential type. Boundary layer theories for fluid similar to a second grade fluid have been formulated by Rajeswari and Rathna, Bhatnagar, Beard and Waiters and Frater. Rajagopal *et al.* developed a boundary layer approximation for a second grade fluid [1].

The third grade fluid models even for steady flow exhibits such characteristics. The present study deals with the problem of non-Newtonian fluid of third grade in a porous half space. Due to the widespread applications, flow through porous media received substantial attention. The attempts to include porous media in the flows of the complex fluids need some new physical parameters besides non-Newtonian fluid parameters. Thus, Darcys equations or some generalization of it depending on pressure field, not neglecting porosity, are appropriate to study this type of flows thorough the porous media which is rigid or nearly rigid solid. Also the modeling of polymeric flow in porous space has essential focus on the numerical simulation of viscoelastic flows in a specific pore geometry models, including: capillary tubes, undulating tubes, packs of spheres or cylinders [2, 3].

Moreover, spectral methods have been successfully applied in the approximation of differential boundary value problems defined in unbounded domains. For problems solutions of which are sufficiently smooth, they exhibit exponential rates of convergence/spectral accuracy. The first approach is using Laguerre polynomials [4-6]. The Burgers equation and Benjamin-Bona-Mahony (BBM) equation on a semi-infinite interval are two equations that Guo [4] worked them out and suggested a Laguerre-Galerkin method for them. It is shown that the Laguerre-Galerkin approximations are convergent on a semi-infinite interval with spectral accuracy. In [5] proposed spectral methods Laguerre functions were used and analyzed for model elliptic equations on regular unbounded domains. Siyyam [6] applied two numerical methods for solving initial value problem differential equations using the Laguerre Tau method. He generated linear systems and solved them. Maday *et al.* [7] proposed a Laguerre type spectral method for solving partial differential equations. They introduced a general presentation of the method and a description of the derivation discretization matrices and then determined the optimum estimations in the adapted Hilbert norms. Recently, in [8] it used the modified generalized Laguerre. However, modified generalized

Corresponding Author: Kourosh Parand, Department of Computer Sciences, Shahid Beheshti University, Tehran, Iran. Tel: +98 21 22431653, Fax: +98 21 22431650.

Laguerre was used befor. The second approach is reformulating original problem in semi-infinite domain to singular problem in bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting singular problem [9]. The third approach is replacing semi-infinite domain with $[0, L]$ interval by choosing L, sufficiently large. This method is named domain truncation [10]. The fourth approach of spectral method is based on rational orthogonal functions [11]. Boyd [12] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials. Guo *et al.* [13] introduced a new set of rational Legendre functions which is mutually orthogonal in $L^2(0, +\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Among these, an approach consists in using the Pseudospectral method based on the nodes of Gauss formulas related to unbounded intervals [14].

Collocation method has become increasingly popular for solving differential equations; although, this is very useful in providing highly accurate solutions to differential equations. Recently, it was used [14-16].

In this paper, we aim to employ the collocation method for solving steady flow of a third grade fluid in a porous half space: In Section 2, we describe the mathematical formulation of this problem. In Section 3, we describe the formulation of modified generalized Laguerre functions. Section 4 summarizes the application of this method for solving steady flow of a third grade fluid in a porous half space. In Section 5, we show results and in this section a comparison is made with Ahmad solutions. Conclusions are described in the final section.

Mathematical Formulation: In this section we focus on Hayat *et al.* [2] who have discussed the flow of a third grade fluid in a porous half space. For unidirectional flow, they have generalized the relation [2].

$$(\nabla p)_x = -\frac{\mu\varphi}{k}\left(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right)u, \tag{1}$$

for a second grade fluid to the following modified Darcy's Law for a third grade fluid

$$(\nabla p)_x = -\frac{\varphi}{k}\left[\mu u+\alpha_1\frac{\partial u}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2 u\right], \tag{2}$$

Where μ is the dynamic viscosity, u is the denote the fluid velocity and p is the pressure, k and φ, respectively represent the permeability and porosity of the porous half space which occupies the region $y > 0$ and a_1, β_3 are material constants. Defining non dimensional fluid velocity f and the coordinate z

$$z = \frac{V_0}{v}y, \qquad f(z) = \frac{u}{V_0}, \qquad V_0 = u(0),\ v = \frac{\mu}{\rho}, \tag{3}$$

Where u and V_0 represent the kinematic viscosity, the boundary value problem modeling the steady state flow of a third grade fluid in a porous half space becomes [2].

$$f''(z) + b_1 f'^2(z) f''(z) - b_2 f(z) f'^2(z) - b_3 f(z) = 0, \tag{4}$$

$$f(0) = 1,\ f(\infty) = 0. \tag{5}$$

Where b_1, b_2 and b_3 are defined as:

$$b_1 = \frac{6\beta_3 V_0^4}{\mu v^2}, \quad b_2 = \frac{2\beta_3\varphi V_0^2}{k\mu}, \quad b_3 = \frac{\varphi v^2}{k V_0^2}. \tag{6}$$

Note that the parameters are not independent, since

$$b_2 = \frac{b_1 b_3}{3}. \tag{7}$$

The homotopy analysis method for solution of Eq. (4) found in [2]. Later Ahmad gave the asymptotic form of the solution and utilize this information to develop a series solution [17].

Modified Generalized Laguerre Functions: This section is devoted to the introduction of the basic notions and working tools concerning orthogonal modified generalized Laguerre. It has been widely used for numerical solutions of differential equations on infinite intervals. $L_n^{\alpha}(x)$ (generalized Laguerre polynomial) is the nth eigenfunction of the Sturm-Liouville problem [14, 18, 19]:

$$x\frac{d^2}{dx^2}L_n^\alpha(x)+(\alpha+1-x)\frac{d}{dx}L_n^\alpha(x)+nL_n^\alpha(x)=0,$$
$$x\in I=[0,\infty),\quad n=0,1,2,.... \tag{8}$$

The generalized Laguerre in polynomial manner are defined with the following recurrence formula:

$$L_0^\alpha(x)=1, \tag{9}$$

$$L_1^\alpha(x)=1+\alpha-x,$$

$$nL_n^\alpha(x)=(2n-1+\alpha-x)L_{n-1}^\alpha(x)-(n+\alpha-1)L_{n-2}^\alpha(x),$$

These are orthogonal polynomials for the weight function $w_a = x^a e^{-x}$. We define Modified generalized Laguerre functions (which we denote MGLF) ϕ_j as follows [14]:

$$\phi_j(x)=\exp(\frac{-x}{2L})L_j^1(\frac{x}{L}),\quad L\succ 0. \tag{10}$$

This system is an orthogonal basis [20, 21] with weight function $w(x)=\frac{x}{L}$ and orthogonality property [14]:

$$\langle\phi_m,\phi_n\rangle_{w_L}=(\frac{\Gamma(n+2)}{L^2 n!})\delta_{nm}, \tag{11}$$

Where δ_{nm} is the Kronecker function.

Function Approximation with Laguerre Functions: A function $f(x)$ defined over the interval $I = [0, \infty)$ can be expanded as

$$f(x)=\sum_{i=0}^{+\infty}a_i\phi_i(x), \tag{12}$$

Where

$$a_i=\frac{\langle f,\phi_i\rangle_w}{\langle\phi_i,\phi_i\rangle_w}. \tag{13}$$

If the infinite series in Eq. (12) is truncated with N terms, then it can be written as [14]

$$f(x)\cong\sum_{i=0}^{N-1}a_i\phi_i(x)=A^T\phi(x), \tag{14}$$

with

$$A=[a_0,a_1,a_2,...,a_{N-1}(x)]^T \tag{15}$$

$$\phi(x)=[\phi_0(x),\phi_1(x),\phi_2(x),...,\phi_{N-1}(x)]^T \tag{16}$$

Modified Generalized Laguerre Functions Collocation Method: Laguerre-Gauss-Radau points and generalized Laguerre-Gauss-type interpolation were introduced by [14, 22-24].

Let

$$R_N=span\{1,x,...,x^{2N-1}\} \tag{17}$$

we choose the collocation points relative to the zeroes of the functions [14]

$$P_j(x)=\phi_j(x)-(\frac{j+1}{j})\phi_{j-1}(x). \tag{18}$$

Let $w(x)=\frac{x}{L}$ and x_j, $j = 0,1,...,N-1$, be the N MGLF-Radau points. The relation between MGLF orthogonal systems and MGLF integrations is as follows [14, 25]:

$$\int_0^{+\infty}f(x)w(x)dx=\sum_{j=0}^{N-1}f_j(x)w_j+\left(\frac{\Gamma(N+2)}{(N)!(2N)!}\right)f^{2N}(\xi)e^{\xi}, \tag{19}$$

Where $0 < \xi < \infty$ and $w_j=x_j\frac{\Gamma(N+2)}{(L(N+1)![(N+1)\phi_{N+1}(x_j)]^2)}$, $j = 0,1,2,...N-1$. In particular, the second term on the right-hand side vanishes when $f(x)$ is a polynomial of degree at most $2N-1$ [14]. We define

$$I_N u(x)=\sum_{j=0}^{N-1}a_j\phi_j(x), \tag{20}$$

it as: $I_N u(x_j) = u(x_j)$, $j = 0,1,2,...,N-1$. $I_N u$ is the orthogonal projection of u upon R_N with respect to the discrete inner product and discrete norm as [14]:

$$\langle u,v\rangle_{w,N}=\sum_{j=0}^{N-1}u(x_j)v(x_j)w_j, \tag{21}$$

$$\|u\|_{w,N}=\langle u,v\rangle_{w,N}^{\frac{1}{2}}, \tag{22}$$

thus for the MGLF Gauss-Radau interpolation we have

$$\langle I_N u,v\rangle_{w,N}=\langle u,v\rangle_{w,N},\quad\forall u.v\in\Re_N. \tag{23}$$

Solving the Problem with Modified Generalized Laguerre Functions: To apply modified generalized Laguerre collocation method to Eq. (4) with boundary conditions Eq. (5), at first we expand $f(z)$ as follows:

$$I_N f(z) = \sum_{j=0}^{N-1} a_j \phi_j(z), \tag{24}$$

To find the unknown coefficients a_j's, we substitute the truncated series $f(z)$ into Eq. (4) and boundary conditions in Eq. (5). Also, we define Residual function of the form

$$\begin{aligned} \mathrm{Re}s(z) = &\sum_{j=0}^{N-1} a_j \phi_j''(z) + b_1 (\sum_{j=0}^{N-1} a_j \phi_j'(z))^2 \sum_{j=0}^{N-1} a_j \phi_j''(z) \\ &- b_2 \sum_{j=0}^{N-1} a_j \phi_j(z) (\sum_{j=0}^{N-1} a_j \phi_j'(z))^2 - b_3 \sum_{j=0}^{N-1} a_j \phi_j(z), \end{aligned} \tag{25}$$

$$\sum_{j=0}^{N-1} a_j \phi_j(0) = 1, \tag{26}$$

$$\sum_{j=0}^{N-1} a_j \phi_j(\infty) = 0, \tag{27}$$

By applying z in Eq. (25) with the N collocation points which are roots of functions L_N^{α}, we have N equations that generates a set of N nonlinear equations; also, we have one boundary equation in Eq. (26). Now, all of these equations can be solved by Newton method for the unknown coefficients. We must mention Eq. (27) is always true; therefore, we do not need to apply this boundary condition.

RESULT AND DISCUSSION

In this paper, we present the results of our research by $N = 20$, $\alpha = 1$ and $L = 0.99$ in modified generalized Laguerre for solving this problem for some typical values of parameters, $b_1 = 0.6$, $b_2 = 0.1$ and $b_3 = 0.5$. In this problem the numerical solution of $f'(0)$ is important. Ahmad [17] obtained $f'(0)$ by the shooting method and founded, correct to six decimal positions, $f'(0) = -0.678301$. We compare the present method with numerical solution and Ahmad solution [17] in Table 1. It shows the present method is highly accurate. Also, the solution is presented graphically in Figure 1.

Table 1: Comparison between MGLF and Ahmad solutions [17] for $b_1 = 0.6$, $b_2 = 0.1$ and $b_3 = 0.5$ with $N = 20$ $\alpha = 1$ and $L = 0.99$

z	Ahmad method[17]	Present method	Numerical[17]
0.0	1.00000	1.00000	1.00000
0.2	0.87220	0.87261	0.87260
0.4	0.76010	0.76063	0.76060
0.6	0.66190	0.66243	0.66240
0.8	0.57600	0.57650	0.57650
1.0	0.50100	0.50144	0.50140
1.2	0.43560	0.43595	0.43590
1.6	0.32890	0.32920	0.32920
2.0	0.24820	0.24838	0.24840
2.5	0.17440	0.17455	0.17450
2.7	0.15140	0.15156	0.15160
3.0	0.12250	0.12261	0.12260
3.4	0.09234	0.09242	0.09242
3.6	0.08016	0.08024	0.08024
4.0	0.06042	0.06047	0.06047
4.2	0.05245	0.05250	0.05250
4.4	0.04553	0.04558	0.04558
4.6	0.03953	0.03957	0.03957
4.8	0.03432	0.03435	0.03435
5.0	0.02979	0.02982	0.02982
$f'(0)$	-0.681835	-0.678297	-0.678301

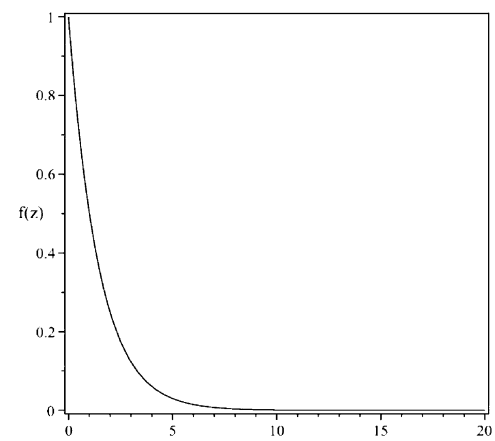

Fig. 1: Graph of numerical approximate $f(z)$ by MGLF with $N = 20$, $\alpha = 1$ and $L = 0.99$.

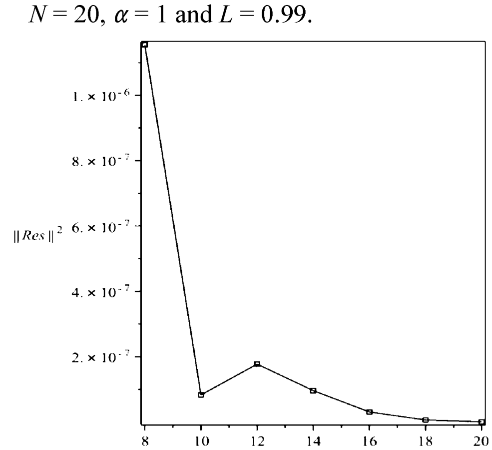

Fig. 2: Graph of $||\mathrm{Re}s||^2$ by MGLFs solution.

Also, the graph of the $||Res||^2$ for MGLF at $b_1 = 0.6$, $b_2 = 0.1$ and $b_3 = 0.5$ is shown in Figure 2. This graph illustrates the convergence rate of the method.

CONCLUSION

In present study, the steady flow of the third grade fluid in a porous half space is considered using modified generalized Laguerre functions method. Modified generalized Laguerre functions are orthogonal functions that solved the system of non-linear differential equations governing the problem on the semi-infinite domain without truncating it to a finite domain. Modified generalized Laguerre functions were proposed to provide simple way to improve the convergence of the solution by collocation method. As a final point, we reported our numerical finding and demonstrated the present solution with MGLF was highly accurate.

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