Comparison between Hermite and Sinc collocation methods for solving steady flow of a third grade fluid in a porous half space

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Abstract—***In this paper, we provide a collocation method for the problem of steady flow of a third grade fluid in a porous half space. This problem is a non-linear, two-point boundary value problem (BVP) on semi-infinite interval. We use two orthogonal functions namely Hermite and Sinc functions which will be defined as basis functions in this approach and compare them together. We also present comparison of these works with numerical solution that shows the present solutions are accurate and applicable.***

Keywords-component: Steady flow, Porous half space, Hermite function, Sinc function, Spectral methods, Semi-infinite.

I. Introduction

A. Introduction of the problem

The flow of non-Newtonian fluids has several technical applications, especially in the paper and textile industries. Out of many models which have been used to describe the non-Newtonian behavior exhibited by certain fluids. The fluids of the differential type have received special attention. Fluids of the second and the third grade have been studied in various types of flow situations which form a subclass of the fluids of the differential type. Boundary layer theories for fluid similar to a second grade fluid have been formulated by Rajeswari and Rathna, Bhatnagar, Beard and Waiters, and Frater. Rajagopal et al. developed a boundary layer approximation for a second grade fluid [1].

The third grade fluid models even for steady flow exhibits such characteristics. The present study deals with the problem of non-Newtonian fluid of third grade in a porous half space. Due to the widespread applications, flow through porous media received substantial attention. The attempts to include porous media in the flows of the complex fluids need some new physical parameters besides non-Newtonian fluid parameters. Thus, Darcys equations or some generalization of it depending on pressure field, not neglecting porosity, are appropriate to study this type of flows thorough the porous media which is rigid or nearly rigid solid. Also the modeling of polymeric flow in porous space has essential focus on the numerical simulation of viscoelastic flows in a specific pore geometry models, including: capillary tubes, undulating tubes, packs of spheres or cylinders [2, 3].

B. Spectral method

Many of the current science and engineering problems are set in unbounded domains. In the context of spectral methods such as collocation and Galerkin methods [4], a number of approaches for treating unbounded domains have been proposed and investigated. The most common method is the use of polynomials that are orthogonal over unbounded domains, such as the Hermite and Laguerre spectral method [5-12].

Guo [13-16] proposed a method that proceeds by mapping the original problem in an unbounded domain to a problem in a bounded domain, and then using suitable Jacobi polynomials such as Gegenbauer polynomials to approximate the resulting problems. The Jacobi polynomials are a class of classical orthogonal polynomials and the Gegenbauer polynomials, and thus also the Legendre and Chebyshev polynomials, are special cases of these polynomials which have been used in sevral literatures for solving some problems [17, 18].

On more approach is replacing infinite domain with $[-L, L]$ and semi-infinite interval with $[0, L]$ by choosing L, sufficiently large. This method is named domain truncation [19].

There is another effective direct approach for solving such problems which is based on rational approximations. Christov [20] and Boyd [21, 22] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [21] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials. Guo et al. [23] introduced a new set of rational Legendre functions which are mutually orthogonal in $L^2(0,\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half-line. Boyd et al. [24] applied pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine methods.

Parand et al. [25-30], applied spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on rational tau and collocation methods.

In this paper, we are going to solve the model Eq. (1) numerically by using two orthogonal functions, namely Hermit function and Sinc function in collocation method and compare our result together. we also have a comparison with solutions of [31].
Sections *III* and *V* review the desirable properties of Hermit function and Sinc function with solution of the problem with collocation method by these functions, respectively. In Section *VII* we describe our results via tables and figures. Finally, concluding remarks will be presented in Section *VIII*.

II. Mathematical formulation

In this section we focus on Hayat et al. [2] who have discussed the flow of a third grade fluid in a porous half space. For unidirectional flow, they have generalized the relation [2]

$$(\nabla p)_x = -\frac{\mu\varphi}{k}(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t})u,$$

for a second grade fluid to the following modified Darcy's Law for a third grade fluid

$$(\nabla p)_x = -\frac{\varphi}{k}[\mu u+\alpha_1\frac{\partial u}{\partial t}+2\beta_3(\frac{\partial u}{\partial y})^2 u],$$

where μ is the dynamic viscosity, u is the denote the fluid velocity and p is the pressure, k and φ, respectively represent the permeability and porosity of the porous half space which occupies the region $y>0$ and α_1, β_3 are material constants. Defining non dimensional fluid velocity f and the coordinate z

$$z=\frac{V_0}{\nu}y, \qquad f(z)=\frac{u}{V_0},$$
$$V_0=u(0), \qquad \nu=\frac{\mu}{\rho},$$

where ν and V_0 represent the kinematic viscosity, the boundary value problem modeling the steady state flow of a third grade fluid in a porous half space becomes [2]

$$f''(z)+b_1 f'^2(z)f''(z)-b_2 f(z)f'^2(z)-b_3 f(z)=0,$$
$$f(0)=1, \quad f(\infty)=0. \tag{1}$$

Where b_1, b_2 and b_3 are defined as:

$$b_1=\frac{6\beta_3 V_0^4}{\mu\nu^2},$$
$$b_2=\frac{2\beta_3\varphi V_0^2}{k\mu},$$
$$b_3=\frac{\varphi\nu^2}{kV_0^2}.$$

Note that the parameters are not independent, since

$$b_2=\frac{b_1 b_3}{3}.$$

The homotopy analysis method for solution of Eq. (1) found in [2]. Later Ahmad gave the asymptotic form of the solution and utilize this information to develop a series solution [31].

III. Hermite function

This section are devoted to elaborate the properties of Hermite functions. First of all, we should mention Hermite polynomials are generally not suitable in practice due to their wild asymptotic behavior at infinities [32]; therefore, we shall consider the Hermite function. The normalized Hermite functions of degree n is defined by [33]

$$\tilde{H}_n=\frac{1}{\sqrt{2^n n!}}e^{\frac{-x^2}{2}}H_n(x), \quad n\ge 0, x\in\Re.$$

That $\{\tilde{H}_n\}$ is an orthogonal system in $L^2(\Re)$.
In the contrary to Hermite Polynomials, the Hermite functions are well behaved with the decay property:

$$|\tilde{H}_n(x)|\to 0, \quad as \ |x|\to\infty,$$

and, the three-term recurrence relation of Hermite functions implies [33]

$$\tilde{H}_{n+1}(x)=x\sqrt{\frac{2}{n+1}}\tilde{H}_n(x)-\sqrt{\frac{n}{n+1}}\tilde{H}_{n-1}(x), \quad n\ge 1,$$
$$\tilde{H}_0(x)=e^{\frac{-x^2}{2}}, \qquad \tilde{H}_1(x)=\sqrt{2}xe^{\frac{-x^2}{2}}.$$

For more details you can study [33-35].
Steady flow problem is defined on the interval $(0,+\infty)$, but Hermite functions are defined on the interval $(-\infty,+\infty)$. One of the approaches to construct an approximation on the interval $(0,+\infty)$ is using mapping that is changing variable of the form [33]

$$w=\Phi(z)=\frac{1}{k}ln(z),$$

where k is a constant.
The transformed Hermite functions are

$$\hat{H}_n(x)\equiv\tilde{H}_n(x)\circ\Phi(x)=\tilde{H}_n(\Phi(x)),$$

The inverse map of $w=\Phi(z)$ is

$$z=\Phi^{-1}(w)=e^{kw}.$$

Therefor, we may define the inverse images of the spaced nodes $\{x_j\}_{x_j=-\infty}^{x_j=+\infty}$ as [33]

$$\Gamma=\{\Phi^{-1}(t):-\infty<t<+\infty\}=(0,+\infty),$$

and

$$\tilde{x}_j = \Phi^{-1}(x_j) = e^{x_j},\ j = 0,1,2,\ldots$$

Let $w(x)$ denotes a non-negative, integrable, real-valued function over the interval Γ, We define [33]

$$L_w^2(\Gamma) = \{v : \Gamma \to \mathrm{R} \mid v \, is \, measurable and \, \|v\|_w < \infty\},$$

where

$$\|v\|_w = (\int_0^\infty |v(x)|^2 \, w(x)dx)^{\frac{1}{2}},$$

is the norm induced by the inner product of the space $L_w^2(\Gamma)$ [33],

$$< u, v >_w = \int_0^\infty u(x)v(x)w(x)dx.$$

Thus, $\{\hat{H}_n(x)\}_{n\in N}$ denotes a system which is mutually orthogonal

$$\langle \hat{H}_n(x), \hat{H}_m(x)\rangle_{w(x)} = \sqrt{\pi}\delta_{nm}.$$

This system is complete in $L_w^2(\Gamma)$. Therefore, for any function $f \in L_w^2(\Gamma)$ the following expansion holds [33]

$$f(x) \cong \sum_{k=-N}^{+N} f_k \hat{H}_k(x),$$

with

$$f_k = \frac{\langle f(x), \hat{H}_k(x)\rangle_{w(x)}}{\left\|\hat{H}_k(x)\right\|_{w(x)}^2}.$$

Now we define an orthogonal projection based on the transformed Hermite function as given below [33].

Let

$$\overline{H}_N = span\{\hat{H}_0(x), \hat{H}_1(x), \ldots, \hat{H}_n(x)\}.$$

The $L^2(\Gamma)$-orthogonal projection $\hat{\xi}_N : L^2(\Gamma) \to \overline{H}_N$ is a mapping in a way that for any $y \in L^2(\Gamma)$ [33],

$$\langle \xi_N y - y, \phi\rangle = 0 \quad \forall \phi \in \overline{H}_N,$$

or equivalently,

$$\hat{\xi}_N y(x) = \sum_{i=0}^{N} \hat{a}_i \hat{H}_i(x).$$

IV. Solving the problem with Hermite function

For solving Steady Flow Problem, we used $\frac{1}{k} ln(z)$ for changing variable. Also, because of boundary conditions, we set following function:

$$p(z) = \frac{1}{1 + \lambda z + z^2},$$

and λ is constant.

Finally, we have

$$\hat{\xi}_N f(z) = p(z) + \hat{\xi}_N f(z).$$

that

$$\hat{\xi}_N f(z) = \sum_{i=0}^{N} \hat{a}_i \hat{H}_i(z).$$

To find the unknown coefficients a_i's, we substitute the truncated series $\hat{\xi}_N f(z)$ into Eq. (1). Also, we define Residual function of the form

$$\begin{aligned} Res(z) &= (p''(z) + \hat{\xi}_N f''(z)) + b_1(p'(z) \\ &+ \hat{\xi}_N f'(z))^2 (p''(z) + \hat{\xi}_N f''(z)) \\ &- b_2(p(z) + \hat{\xi}_N f(z))(p'(z) + \hat{\xi}_N f'(z))^2 \\ &- b_3(p(z) + \hat{\xi}_N f(z)) = 0. \end{aligned} \tag{2}$$

By applying z in Eq. (2) with the N collocation points which are roots of transformed Hermite function, we have N equations that generates a set of N nonlinear equations. Now, all of these equations can be solved by Newton method for the unknown coefficients.

V. Sinc function

The Sinc function is defined by [36]

$$Sinc(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

For each integer k and the mesh size h, the sinc functions are defined on $\Re$ by [37]

$$S_k(h,x) \equiv Sinc(\frac{x-kh}{h}) = \begin{cases} \dfrac{\sin(\frac{\pi}{h}(x-kh))}{\frac{\pi}{h}(x-kh)} & x \neq kh \\ 1 & x = kh \end{cases}$$

Steady flow problem is defined on the interval $(0,+\infty)$, but Sinc functions are defined on the interval $(-\infty,+\infty)$. One of the approaches to construct an approximation on the interval $(0,+\infty)$ is using mapping that is changing variable of the form

$$w = \Phi(z) = \ln(\sinh(x)),$$

The basis functions on $(0,+\infty)$ are taken to be composite translates Sinc functions [36]:

$$S_k(x) \equiv S(k,h) \circ \Phi(x) = Sinc(\frac{\Phi(x) - kh}{h}),$$

Where $S(k,h) \circ \Phi(x)$ is defines by $S(k,h)(\Phi(x))$. The inverse map of $w = \Phi(z)$ is [38]

$$z = \Phi^{-1}(w) = \ln(e^w + \sqrt{e^{2w} + 1}).$$

Thus,

$$x_k = \Phi^{-1}(kh) = \ln(e^{kh} + \sqrt{e^{2kh} + 1}), k = 0,\pm1,\pm2,\ldots$$

Let $w(x)$ denotes a non-negative, integrable, real-valued function over the interval $(0,+\infty)$. We define [36]:

$$L_w^2(\Gamma) = \left\{ v : \Gamma \to \Re \middle| v \quad is \ measurable \ and \ \|v\|_w \prec \infty \right\},$$

Where

$$\|v\|_w = (\int_0^\infty |v(x)|^2 w(x)dx)^{\frac{1}{2}},$$

is the norm induced by the inner product of the space $L_w^2(\Gamma)$:

$$\langle u,v\rangle_w = \int_0^\infty u(x)v(x)w(x)dx.$$

Thus, $\{S_k(x)\}_{k\in z}$ with constant h denotes a system which is mutually orthogonal [36]:

$$\langle S_{k_n}(x), S_{k_m}(x)\rangle_{w(x)} = hS_{nm}.$$

Now, for any function $f \in L_w^2(\Gamma)$, the following expansion holds [36]:

$$f(x) \cong \sum_{k=-N}^{+N} f_k S_k(x).$$

In addition, the nth derivation of the function f at some point x_k can be approximated [36, 39]

$$\delta_{k,j}^{(0)} = [S(k,h)\circ\Phi(x)]|_{x=x_j} = \begin{cases} 1 & k=j \\ 0 & k\neq j \end{cases},$$

$$\delta_{k,j}^{(1)} = \frac{d}{d\Phi}[S(k,h)\circ\Phi(x)]|_{x=x_j} = \frac{1}{h}\begin{cases} 0 & k=j \\ \frac{(-1)^{j-k}}{j-k} & k\neq j \end{cases},$$

$$\delta_{k,j}^{(2)} = \frac{d^2}{d\Phi^2}[S(k,h)\circ\Phi(x)]|_{x=x_j} = \frac{1}{h^2}\begin{cases} \frac{-\pi^2}{3} & k=j \\ \frac{-2(-1)^{j-k}}{(j-k)^2} & k\neq j \end{cases}.$$

VI. Solving the problem with Sinc function

For solving Steady Flow Problem, we used $\ln(\sinh(z))$ for changing variable. Also, because of boundary conditions, we set following function:

$$p(z) = \frac{1}{1+\lambda z+z^2},$$

and λ is constant.

Finally, we have

$$f(z) \cong f_N(z) = p(z) + u_N(z),$$

that

$$u_N(z) = \sum_{k=-N}^{+N} c_k \frac{zS_k(z)}{z^2+1}.$$

The collocation points are

$$z_j = \ln(e^{jh} + \sqrt{1+e^{2jh}}),\ j=-N,\ldots,+N,$$

And the derivations of $u_N(z)$ are

$$u_N(z_j) = \frac{c_j z_j}{z_j^2+1},$$

$$u'_N(z_j) = \sum_{k=-N}^{+N} c_k \left\{ \left(\frac{1}{1+z_j^2} - \frac{2z_j^2}{(1+z_j^2)^2}\right)\delta_{k,j}^{(0)} + \left(\frac{z_j\Phi'(z_j)}{1+z_j^2}\right)\delta_{k,j}^{(1)} \right\},$$

$$u''_N(z_j) = \sum_{k=-N}^{+N} c_k \left\{ \left(\frac{-6z_j}{(1+z_j^2)^2} + \frac{8z_j^3}{(1+z_j^2)^3}\right)\delta_{k,j}^{(0)} \right.$$

$$\left. + \left(\frac{2\Phi'(z_j)}{1+z_j^2} - \frac{4z_j^2\Phi'(z_j)}{(1+z_j^2)^2} + \frac{z_j\Phi'(z_j)}{1+z_j^2}\right)\delta_{k,j}^{(1)} + \left(\frac{z_j(\Phi'(z_j))^2}{1+z_j^2}\right)\delta_{k,j}^{(2)} \right\}.$$

To find the unknown coefficients c_i's, we substitute the truncated series $f_N(z_j)$ into Eq. (1). Also, we define Residual functions of the form

$$f''_N(z_j) + b_1(f'_N(z_j))^2 f''_N(z_j)$$
$$-b_2 f_N(z_j)(f'_N(z_j))^2 - b_3 f_N(z_j) = 0,\ j=-N,\ldots,+N.$$

We have $2N+1$ nonlinear equations. Now, all of these equations can be solved by Newton method for the unknown coefficients.

VII. Result

In this paper, we present the results of our research about Hermite function by $N=16, k=1.2,$ and $\lambda = 0.678301$ and Sinc function by $N=17,\ h=1,$ and $\lambda = 0.47$ for solving this problem for some typical values of parameters, $b_1 = 0.6,\ b_2 = 0.1,$ and $b_3 = 0.5$. In this problem the numerical solution of $f'(0)$ is important. Ahmad [31] obtained $f'(0)$ by the shooting method and founded correct to six decimal positions $f'(0) = -0.678301$.

We compare the present methods with numerical solution and Ahmad solution [31], also we compare them with each other in Table 1. Also, the solutions are presented graphically in Figure 1 and Figure 2.

Table 1. Comparison between Hermite function, Sinc Function, Ahmad method [31], and Shooting method [31].

Shooting [31]	Ahmad [31]	Sinc function	Hermite function	z
1.00000	1.00000	1.00000	1.00000	0.0
0.87260	0.87220	0.87278	0.87261	0.2
0.76060	0.76010	0.76035	0.76064	0.4
0.66240	0.66190	0.66178	0.66243	0.6
0.57650	0.57600	0.57597	0.57647	0.8
0.50140	0.50100	0.50115	0.50139	1.0
0.43590	0.43560	0.43583	0.43591	1.2
0.32920	0.32890	0.32905	0.32917	1.6
0.24840	0.24820	0.24802	0.24839	2.0
0.17450	0.17440	0.17426	0.17459	2.5
0.15160	0.15140	0.15141	0.15161	2.7
0.12260	0.12250	0.12265	0.12270	3.0
0.08024	0.08016	0.08025	0.08036	3.6
0.06047	0.06042	0.06033	0.06060	4.0
0.05250	0.05245	0.05233	0.05261	4.2
0.04558	0.04553	0.04543	0.04567	4.4
0.03957	0.03953	0.03948	0.03964	4.6
0.03435	0.03432	0.03434	0.03440	4.8
0.02982	0.02979	0.02987	0.02984	5.0
-0.678301	-0.681835	-0.677843	-0.678301	$f'(0)$

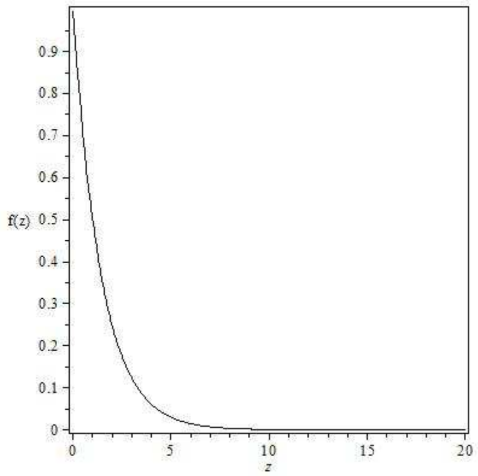

Figure 1. Graph of numerical approximate $f(z)$ by Hermite function

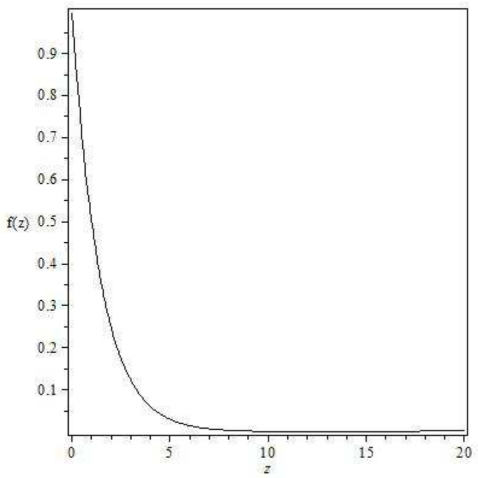

Figure 2. Graph of numerical approximate $f(z)$ by Sinc function

VIII. Conclusions

In this paper, we applied the collocation method to solve the steady flow of the third grade fluid in a porous half space. This method is easy to implement and yields the desired accuracy. An important concern of collocation approach is the choice of basis functions. The basis functions have three different properties: easy computation, rapid convergence and completeness, which means that any solution can be represented to arbitrarily high accuracy by taking the truncation N to be sufficiently large. We used two set of orthogonal functions as the basis function in this method and compared the results together. Through the comparisons among the numerical solutions [31] and the current works, it has been shown that the present works have provided acceptable approach for this type equation. Although both functions lead to more accurate results, but it seems that the accuracy and rapidity of Hermite function is better than Sinc function in this problem.

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