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Research Article

Comparison between Rational Gegenbauer and Modified Generalized Laguerre Functions Collocation Methods for Solving the Case of Heat Transfer Equations Arising in Porous Medium

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Abstract
In this paper, we provide the collocation method for natural convection heat transfer equations embedded in porous medium which are of great importance in the design of canisters for nuclear waste disposal. This problem is a non-linear, three-point boundary value problem on semi-infinite interval. We use two orthogonal functions namely rational Gegenbauer and modified generalized Laguerre functions which are defined as basis functions in this approach and compare them together. We also present the comparison of these works with Runge-Kutta solution, moreover, in the graph of the $\|Res\|^2$, we show that the present solutions are accurate and applicable.
Keywords : Rational Gegenbauer; Modified generalized Laguerre; Collocation method; Nonlinear ODE; Indirect multiquadric; Porous media.

1 Introduction

1.1 Introducing of the problem

Natural convective heat transfer in porous media has received considerable attention during the past few decades. This interest can be attributed to its wide range of applications in ceramic processing, nuclear reactor cooling system, crude oil drilling, chemical reactor design, ground water pollution and filtration processes. External natural convection in a porous medium adjacent to heated bodies was analyzed by Nield and Bejan [39], Merkin [33, 34], Minkowycz and Cheng [35, 36, 37], Pop and Cheng [12, 47], Ingham and Pop [29]. All through these studies, it is assumed that the boundary layer approximations are applicable and the coupled set of governing equations are solved by numerical methods. Also, [1, 51] worked out this problem. Parand [40] Compared two common collocation approaches based on radial basis functions for the case of heat transfer equations arising in porous medium.
In present study, we consider the problem of natural convection about an inverted heated cone

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embedded in a porous medium of infinite extent. No similarity solution exists for the truncated cone, but for the case of full cone similarity solutions exist if the prescribed wall temperature or surface heat flux is a power function of distance from the vertex of the inverted cone [12, 39, 54, 55]. Bejan and Khair [3] used Darcy's law to study the vertical natural convective flows driven by temperature and concentration gradients. Nakayama and Hossain [38] applied the integral method to obtain the heat and mass transfer by free convection from a vertical surface with constant wall temperature and concentration. Yih [60] examined the coupled heat and mass transfer by free convection over a truncated cone in porous media for variable wall temperature and variable heat and mass fluxes, Also he [61] applied the uniform transpiration effect on coupled heat and mass transfer in mixed convection about inclined surfaces in porous media for the entire regime. Cheng [10] used an integral approach to study the heat and mass transfer by natural convection from truncated cones in porous media with variable wall temperature and [11] studies the Soret and Dufour effects on the boundary layer flow due to natural convection heat and mass transfer over a vertical cone in a porous medium, saturated with Newtonian fluids with constant wall temperature. Natural convective mass transfer from upward-pointing vertical cones, embedded in saturated porous media, was studied using the limiting diffusion [50]. The natural convection along with an isothermal wavy cone embedded in a fluid-saturated porous medium were presented in [48, 49]. In [54, 55] fluid flow and heat transfer of vertical full cone embedded in porous media were solved by Homotopy analysis method.
If we want to express the problem formulation of this model, we can consider an inverted cone with semi-angle γ and take axes in the manner indicated in Fig. 1 (a). The boundary layer develops over the heated frustum $x = x_0$. In terms of the stream function ψ is defined by:

$$u = \frac{1}{r}\frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{1.1}$$

The boundary layer equations for natural convection of Darcian fluid about a cone are:

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{1.2}$$

$$u = \frac{\rho_\infty \beta K g \cos\gamma (T - T_\infty)}{\mu} \tag{1.3}$$

$$\frac{1}{r}\left(\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y}\right) = \alpha \frac{\partial^2 T}{\partial y^2}. \tag{1.4}$$

For a thin boundary layer, r is obtained approximately $x\sin(\gamma)$. Suppose that a power law of heat flux is prescribed on the frustum. Accordingly, the boundary conditions at infinity are:

$$u = 0, \quad T = T_\infty, \quad y \to \infty, \tag{1.5}$$

and at the wall are

$$v = 0, \quad y = 0. \tag{1.6}$$

The surface heat flux q_w is prescribed by

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = A(x - x_0)^\lambda \qquad x_0 \le x \le \infty. \tag{1.7}$$

For the case of a full cone ($x_0 = 0$, Fig.1(b)) a similarity solution exists.
In the case of prescribed surface heat flux, we let:

$$\begin{aligned} \psi &= \alpha r (Ra_x)^{1/3} f(\eta), \\ T - T_\infty &= \frac{q_w x}{k}(Ra_x)^{-\frac{1}{3}}\theta(\eta), \\ \eta &= \frac{y}{x}(Ra_x)^{1/3}, \end{aligned} \tag{1.8}$$

where

$$Ra_x = \frac{\rho_\infty \beta g K \cos(\gamma) q_w x^2}{\mu \alpha k}, \tag{1.9}$$

is the local Rayleigh number for the case of prescribed surface heat flux. The governing equations become

$$\begin{aligned} &f' = \theta \\ &\theta'' + \frac{\lambda+5}{2} f\theta' - \frac{2\lambda+1}{3} f'\theta = 0. \end{aligned} \tag{1.10}$$

Subjected to boundary conditions as:

$$f(0) = 0, \quad \theta'(0) = -1, \quad \theta(\infty) = 0. \tag{1.11}$$

Finally from Equations (1.10) and (1.11) we have:

$$\begin{cases} ODE. \quad f''' + \left(\frac{\lambda+5}{2}\right) ff'' - \left(\frac{2\lambda+1}{3}\right)(f')^2 = 0, \\ B.C. \quad f(0) = 0, \quad f''(0) = -1, \quad f'(\infty) = 0. \end{cases} \tag{1.12}$$

It is of interest to obtain the value of the local Nusselt number which is defined as:

$$Nu_x = \frac{q_w x}{k(T_w - T_\infty)}. \tag{1.13}$$

From Eqs. (1.13), (1.8) and (1.9) it follows that the local Nusselt number which is of interest to obtain given by:

$$Nu_x = Ra_x^{1/3}[-\theta(0)]. \tag{1.14}$$

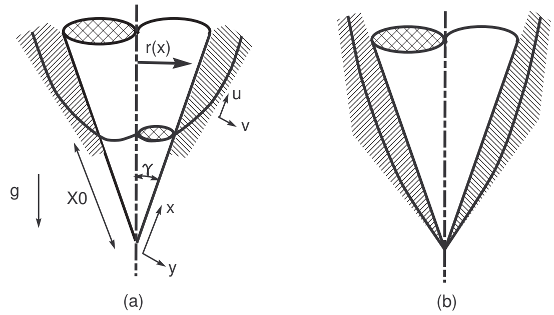

Fig. 1. (a) Coordinate system for the boundary layer on a heated frustum of a cone, (b) full cone, $x_0 = 0$.

1.2 Spectral method

Many of the current science and engineering problems are set in unbounded domains. In the context of spectral methods such as collocation and Galerkin methods [31], a number of approaches for treating unbounded domains have been proposed and investigated. The most common method is the use of polynomials that are orthogonal over unbounded domains, such as the Hermite and Laguerre spectral method [14, 15, 16, 19, 24, 32, 52, 53].

Guo [20, 21, 22, 23] proposed a method that proceeds by mapping the original problem in an unbounded domain to a problem in a bounded domain, and then using suitable Jacobi polynomials such as Gegenbauer polynomials to approximate the resulting problems. The Jacobi polynomials are a class of classical orthogonal polynomials and the Gegenbauer polynomials, thus the Legendre

and Chebyshev polynomials, are special cases of these polynomials also which have been used in sevral literatures for solving some problems [2, 30].
One more approach is replacing infinite domain with $[-L, L]$ and semi-infinite interval with $[0, L]$ by choosing L, sufficiently large. This method is named domain truncation [4].
There is another effective direct approach for solving such problems which is based on rational approximations. Christov [13] and Boyd [5, 6] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [5] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials. Guo et al. [25] introduced a new set of rational Legendre functions which are mutually orthogonal in $L^2(0, \infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half-line. Boyd et al. [8] applied pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine methods.
Parand et al. [41, 42, 43, 44, 45, 46], applied spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on rational tau and collocation methods.
In this paper, solve the model Eq. (1.12) numerically by using two orthogonal functions, namely rational Gegenbauer functions and modified generalized Laguerre functions in collocation method and compare our the obtained results. we also have a comparison with solutions of [55].
Sections 2 and 3 review the desirable properties of rational Gegenbauer functions and modified generalized Laguerre functions with solution of the problem with collocation method by these functions, respectively. In Section 4 we show our results via tables and figures. Finally, concluding remarks are presented in Section 5.

2 Rational Gegenbauer functions

In this section, firstly, rational Gegenbauer functions are introduced and some basic properties of them are presented. Then we approximate a function using Gauss integration with rational Gegenbauer-Gauss points.
The Gegenbauer polynomials $G_n^\alpha(y)$ of order α and of degree n are defined as follows [56, 57]:

$$G_n^\alpha(y) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{\Gamma(n+\alpha-j)}{j!(n-2j)!\Gamma(\alpha)} (2y)^{n-2j}, \tag{2.15}$$

where n is an integer, α is a real number greater than $-\frac{1}{2}$ and Γ is the Gamma function.
The Gegenbauer polynomials are orthogonal in the interval $[-1, 1]$ with respect to the weight function $\rho(y) = (1-y^2)^{\alpha-\frac{1}{2}}$ where $\alpha > -\frac{1}{2}$.
The new basis functions, is denoted by $RG_n^\alpha(x) = G_n^\alpha(y)$, where L is a constant parameter and $y = \frac{x-L}{x+L}$, $y \in [-1, 1]$, the constant parameter L sets the length scale of the mapping . Boyd [7] offered guideline for optimizing the map parameter L.
$RG_n^\alpha(x)$ is the nth eigenfunction of the singular Sturm-Liouville problem:

$$\begin{aligned}&(x+L)\frac{\sqrt{x}}{L}\frac{d}{dx}\left[(x+L)\sqrt{x}\frac{d}{dx}RG_n^\alpha(x)\right] + \alpha\left(\frac{x^2-L^2}{L}\right)\frac{d}{dx}RG_n^\alpha(x)\\ &+n(n+2\alpha)RG_n^\alpha(x) = 0,\end{aligned} \tag{2.16}$$

and satisfies in the following recurrence relation:

$$
\begin{aligned}
&RG_0^\alpha(x) = 1, \qquad RG_1^\alpha(x) = 2\alpha\frac{x-L}{x+L},\\
&RG_{n+1}^\alpha(x) = \frac{1}{n+1}\left[2\left(\frac{x-L}{x+L}\right)(n+\alpha)RG_n^\alpha(x) - (n+2\alpha-1)RG_{n-1}^\alpha(x)\right], n \geq 1.
\end{aligned}
\tag{2.17}
$$

The special cases of rational Gegenbauer functions are rational Legendre functions and rational Chebyshev functions, that were introduced by Guo [25, 26].

2.1 Function approximation with rational Gegenbauer functions

We determine $w(x) = \frac{2L}{(x+L)^2}\left[1-\left(\frac{x-L}{x+L}\right)^2\right]^{\alpha-\frac{1}{2}}$ as a non-negative, integrable and real-valued weight function for rational Gegenbauer over the interval $I = [0, \infty)$.
Let us denote

$$\rho(y) = (1-y^2)^{\alpha-\frac{1}{2}}, \qquad y = \frac{x-L}{x+L}, \tag{2.18}$$

hence we have

$$\frac{dy}{dx} = \frac{2L}{(x+L)^2}, \qquad \frac{dx}{dy} = \frac{2L}{(y-1)^2}, \qquad w(x)\frac{dx}{dy} = \rho(y). \tag{2.19}$$

Now we define

$$L_w^2(I) = \{v : I \to R \mid v \text{ is measurable and } \| v \|_w < \infty\}, \tag{2.20}$$

where

$$\| v\|_w = (\int_0^\infty \mid v(x) \mid^2 w(x)dx)^{\frac{1}{2}}, \tag{2.21}$$

is the norm induced by the scalar product

$$< u, v >_w = \int_0^\infty u(x)v(x)w(x)dx. \tag{2.22}$$

Thus $\{RG_n^\alpha(x)\}_{n\geq 0}$ denotes a system which is mutually orthogonal under Eq.(2.22), i.e.,

$$< RG_n^\alpha, RG_m^\alpha >_w = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}\delta_{nm}, \tag{2.23}$$

where δ_{nm} is the Kronecker delta function. This system is complete in $L_w^2(I)$. For any function $u \in L_w^2(I)$ the following expansion holds

$$u(x) = \sum_{k=0}^\infty a_k RG_k^\alpha(x), \tag{2.24}$$

with

$$a_k = \frac{< u, RG_k^\alpha >_w}{\| RG_k^\alpha\|_w^2}. \tag{2.25}$$

The a_k's are the expansion coefficients associated with the family $\{RG_k^\alpha(x)\}$.

2.2 Rational Gegenbauer interpolation approximation

Authors of [9, 18] introduced Gauss integration. Later, Guo introduced rational Legendre-Gauss points [25] and rational Chebyshev-Gauss points [26]. Now we want to define rational Gegenbauer-Gauss interpolation. Let

$$\mathcal{RG}_N^{\alpha} = span\{RG_0^{\alpha}, RG_1^{\alpha}, ..., RG_N^{\alpha}\}, \tag{2.26}$$

and y_j, $j = 0, 1, \ldots, N$, be the $N+1$ roots of the polynomial $G_{N+1}^{\alpha}(y)$. These points are known as Gegenbauer-Gauss points. The corresponding Christoffel numbers of them are [57]:

$$\frac{2^{2-2\alpha}\pi\Gamma(N+1+2\alpha)}{(N+1)!\Gamma^2(\alpha)} \times \frac{1}{(1-{y_j}^2)[\frac{d}{dy}G_{N+1}^{\alpha}(y_j)]^2}. \tag{2.27}$$

We define

$$x_j = L\frac{1+y_j}{1-y_j} \quad j = 0, 1, \ldots, N, \tag{2.28}$$

which are called as rational Gegenbauer-Gauss nodes. In fact, these points are zeros of the function $RG_{N+1}^{\alpha}(x)$. Using Gauss integration we have:

$$\begin{aligned} \int_0^{\infty} u(x)w(x)dx &= \int_{-1}^{1} u\left(L\frac{1+y}{1-y}\right)\rho(y)dy \\ &= \sum_{j=0}^{N} u(x_j)w_j \quad \forall u \in \mathcal{RG}_{2N}^{\alpha}, \end{aligned} \tag{2.29}$$

where

$$w_j = \frac{2^{2-2\alpha}\pi\Gamma(N+1+2\alpha)}{(N+1)!\Gamma^2(\alpha)} \times \frac{L}{x_j(x_j+L)^2[\frac{d}{dx}RG_{N+1}^{\alpha}(x_j)]^2}, \tag{2.30}$$

are the corresponding weights with the $N+1$ rational Gegenbauer-Gauss nodes, that can be obtained from Eqs. (2.27) and (2.28).
The interpolating function of a smooth function u on a semi-infinite interval is denoted by $P_N u$. It is an element of $\mathcal{RG}_N^{\alpha}$ and is defined as

$$P_N u(x) = \sum_{k=0}^{N} a_k RG_k^{\alpha}(x). \tag{2.31}$$

$P_N u$ is the orthogonal projection of u upon $\mathcal{RG}_N^{\alpha}$ with respect to the inner product Eq.(2.22) and the norm Eq.(2.21). Thus by the orthogonality of rational Gegenbauer functions, we have [20]

$$< P_N u - u, RG_i^{\alpha} >_w = 0 \quad \forall RG_i^{\alpha} \in \mathcal{RG}_N^{\alpha}. \tag{2.32}$$

To apply a collocation method, we consider the residual $Res(x)$, when the expansion is substituted into the governing equation. The a_k's have to be selected so that the boundary conditions are satisfied, but the residual zero is made at as many (suitable chosen) spatial points as possible.

2.3 Solving the problem with rational Gegenbauer functions

In this part, we use rational Gegenbauer collocation (RGC) method by applying P_N operator on the function $f(\eta)$ under Eq. (2.31). Then, we construct the residual function by substituting $f(\eta)$

by $P_N f(\eta)$ in the model Eq. (1.12). By equalizing $Res(\eta)$ to zero at rational Gegenbauer-Gauss points $(\eta_j, j = 1, 2, ..., N-1)$ plus two boundary conditions, we can find the coefficients a_k:

$$\begin{aligned} Res(\eta) &= \frac{d^3}{d\eta^3} P_N f(\eta) + \left(\frac{\lambda+5}{2}\right) P_N f(\eta) \left(\frac{d^2}{d\eta^2} P_N f(\eta)\right) \\ &\quad - \left(\frac{2\lambda+1}{3}\right) \left(\frac{d}{d\eta} I_N f(\eta)\right)^2, \\ P_N f(0) &= 0, \\ \frac{d^2}{d^2\eta} P_N f(0) &= -1. \end{aligned} \tag{2.33}$$

We note that the third boundary condition is already satisfied. These $N+1$ equations generate a set of $N+1$ nonlinear equations which can be solved by a well-known method such as the Newton method.

3 Modified generalized Laguerre functions

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal modified generalized Laguerre. It has been widely used for numerical solutions of differential equations on infinite intervals. $L_n^\alpha(x)$ (generalized Laguerre polynomial) is the nth eigenfunction of the Sturm-Liouville problem [14, 27, 46]:

$$\begin{aligned} x\frac{d^2}{dx^2} L_n^\alpha(x) + (\alpha + 1 - x)\frac{d}{dx} L_n^\alpha(x) + n L_n^\alpha(x) = 0, \\ x \in I = [0, \infty), \qquad n = 0, 1, 2, ... \end{aligned} \tag{3.34}$$

The generalized Laguerre in polynomial manner is defined applying the following recurrence formula:

$$\begin{aligned} L_0^\alpha(x) &= 1, \\ L_1^\alpha(x) &= 1 + \alpha - x, \\ n L_n^\alpha(x) &= (2n - 1 + \alpha - x) L_{n-1}^\alpha(x) - (n + \alpha - 1) L_{n-2}^\alpha(x). \end{aligned} \tag{3.35}$$

These are orthogonal polynomials for the weight function $w_\alpha = x^\alpha e^{-x}$. We define Modified generalized Laguerre functions (which we denote MGLF) ϕ_j as follows [46]:

$$\phi_j(x) = \exp(\frac{-x}{2L}) L_j^1(\frac{x}{L}), \quad L > 0. \tag{3.36}$$

This system is an orthogonal basis [17, 58] with weight function $w(x) = \frac{x}{L}$ and orthogonality property [46]:

$$< \phi_m, \phi_n >_{w_L} = (\frac{\Gamma(n+2)}{L^2 n!}) \delta_{nm}, \tag{3.37}$$

where δ_{nm} is the Kronecker function.

3.1 Function approximation with Laguerre functions

A function $f(x)$ defined over the interval $I = [0, \infty)$ can be expanded as

$$f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x), \tag{3.38}$$

where

$$a_i = \frac{< f, \phi_i >_w}{< \phi_i, \phi_i >_w}. \quad (3.39)$$

If the infinite series in Eq. (3.38) is truncated with N terms, then it can be written as [46]

$$f(x) \simeq \sum_{i=0}^{N-1} a_i \phi_i(x) = A^T \phi(x), \quad (3.40)$$

with

$$A = [a_0, a_1, a_2, ..., a_{N-1}]^T, \quad (3.41)$$

$$\phi(x) = [\phi_0(x), \phi_1(x), ..., \phi_{N-1}(x)]^T. \quad (3.42)$$

3.2 Modified generalized Laguerre functions collocation method

Laguerre-Gauss-Radau points and generalized Laguerre-Gauss-type interpolation were introduced by [28, 46, 59, 62].
Let

$$\Re_N = span\{1, x, ..., x^{2N-1}\}, \quad (3.43)$$

we choose the collocation points relative to the zeroes of the functions [46]

$$p_j(x) = \phi_j(x) - (\frac{j+1}{j})\phi_{j-1}(x). \quad (3.44)$$

Let $w(x) = \frac{x}{L}$ and x_j, $j = 0, 1, ..., N-1$, be the N MGLF-Radau points. The relation between MGLF orthogonal systems and MGLF integrations is as follows [46, 57]:

$$\int_0^\infty f(x)w(x)\mathrm{d}x = \sum_{j=0}^{N-1} f_j(x)w_j + (\frac{\Gamma(N+2)}{(N)!(2N)!})f^{2N}(\xi)e^{\xi}, \quad (3.45)$$

where $0 < \xi < \infty$ and $w_j = x_j \frac{\Gamma(N+2)}{(L(N+1)![(N+1)\phi_{N+1}(x_j)]^2)}$, $j = 0, 1, 2, ..., N-1$. In particular, the second term on the right-hand side vanishes when $f(x)$ is a polynomial of degree at most $2N-1$ [46]. We define

$$I_N u(x) = \sum_{j=0}^{N-1} a_j \phi_j(x), \quad (3.46)$$

it as: $I_N u(x_j) = u(x_j)$, $j = 0, 1, 2, ..., N-1$. $I_N u$ is the orthogonal projection of u upon $\Re_N$ with respect to the discrete inner product and discrete norm as [46]:

$$< u, v >_{w,N} = \sum_{j=0}^{N-1} u(x_j)v(x_j)w_j, \quad (3.47)$$

$$\| u\|_{w,N} =< u, v >_{w,N}^{\frac{1}{2}}, \quad (3.48)$$

thus for the MGLF Gauss-Radau interpolation we have

$$< I_N u, v >_{w,N} =< u, v >_{w,N}, \quad \forall u.v \in \Re_N. \quad (3.49)$$

3.3 Solving the problem with modified generalized Laguerre functions

To apply modified generalized Laguerre collocation method to Eq. (1.12), first we expand $f(\eta)$ as follows:

$$I_N f(\eta) = \sum_{j=0}^{N-1} a_j \phi_j(\eta), \tag{3.50}$$

to find the unknown coefficients a_j's, we substitute the truncated series $f(\eta)$ into Eq. (1.12) and boundary conditions in it. Also, we define Residual function of the form

$$\begin{aligned} Res(\eta) &= \textstyle\sum_{j=0}^{N-1} a_j \phi_j'''(\eta) + (\frac{\lambda+5}{2}) \sum_{j=0}^{N-1} a_j \phi_j(\eta) \sum_{j=0}^{N-1} a_j \phi_j''(\eta) \\ &-(\tfrac{2\lambda+2}{3})(\textstyle\sum_{j=0}^{N-1} a_j \phi_j'(\eta))^2, \end{aligned} \tag{3.51}$$

$$\sum_{j=0}^{N-1} a_j \phi_j(0) = 0, \qquad \sum_{j=0}^{N-1} a_j \phi_j''(0) = -1, \tag{3.52}$$

$$\sum_{j=0}^{N-1} a_j \phi_j'(\infty) = 0. \tag{3.53}$$

By applying η in Eq. (3.51) with the N collocation points which are roots of functions L_α^N , we have N equations that generates a set of N nonlinear equations; also, we have two boundary equations in Eq. (3.52). Now, all of these equations can be solved by Newton method for the unknown coefficients. We must mention Eq. (3.53) is always true; therefore, we do not need to apply this boundary condition.

4 Result and discussion

In the following tables and figures we make a comparison between Runge-Kutta solution obtained by the MATLAB software command ODE45 [55] and the obtained results of two presented methods in this paper.
Table 1 shows a good agreement between rational Gegenbauer collocation method by $N = 13$ and Runge-Kutta method for $f'(0)$ with various λ. The results for $f'(0)$ with modified generalized Laguerre collocation method by the same N have been shown in Table 2 and comparison has been made between the Runge-Kutta solution. Absolute errors in these two tables show that the presented methods give us an approximate solution with a high degree of accuracy with the small Ns.
Some of the computed results for the variations with η of the functions f' for $\lambda = 1/4$ and $\lambda = 3/4$ are listed in Tables 3 and 4, respectively. In these Tables we compare the result of two methods together and with results presented by [55].

Table 1. Comparison of $f'(0)$ for various λ between rational Gegenbauer collocation (RGC) method and Runge-Kutta solution.

λ	Runge-Kutta solution [55]	α	L	RGC	Absolute Error
0	0.94760	0.1	1.965	0.94761	0.00001
1/4	0.91130	0.1	1.984	0.91129	0.00001
1/3	0.90030	0.1	1.986	0.90031	0.00001
1/2	0.87980	0.1	1.997	0.87980	0.00000
3/4	0.85220	0.1	2.015	0.85215	0.00005
1	0.82760	0.1	2.025	0.82760	0.00000

Table 2. Comparison of $f'(0)$ for various λ between modified generalized Laguerre functions

collocation method and Runge-Kutta solution.

λ	Runge-Kutta solution [55]	α	L	MGLF	Absolute Error
0	0.94760	1	1.2985	0.94770	0.0001
1/4	0.91130	0.94869	1.2493	0.91129	0.00001
1/3	0.90030	1	1.15	0.90031	0.00001
1/2	0.87980	1	1.09	0.87933	0.00047
3/4	0.85220	0.04	1.0394	0.85229	0.00009
1	0.82760	0.655	1.115	0.82760	0.00000

Table 3. The comparison of $f'(\eta)$ for $\lambda = 1/4$ for present methods and RungeKutta solution.

η	Runge-Kutta solution [55]	RGC	Error (RGC)	MGLF	Error (MGLF)
0	0.911295	0.911288	0.000007	0.911292	0.000003
0.1	0.813604	0.813597	0.000007	0.813604	0.000000
0.2	0.721351	0.721339	0.000012	0.721433	0.000082
0.3	0.635531	0.635467	0.000064	0.635728	0.000197
0.4	0.556661	0.556579	0.000082	0.557013	0.000352
0.5	0.484997	0.484953	0.000044	0.485487	0.000490
0.6	0.420587	0.420591	0.000004	0.421104	0.000517
0.7	0.363276	0.363284	0.000008	0.363641	0.000365
0.8	0.312677	0.312664	0.000013	0.312746	0.000069
0.9	0.268264	0.268258	0.000006	0.267986	0.000278
1	0.229508	0.229537	0.000029	0.228879	0.000629
1.1	0.195878	0.195947	0.000069	0.194918	0.000900
1.2	0.166847	0.166936	0.000089	0.165591	0.001256
1.3	0.141837	0.141976	0.000139	0.140399	0.001438
1.4	0.120362	0.120571	0.000209	0.118863	0.001499
1.5	0.102025	0.102266	0.000241	0.100533	0.001492

Table 4. The comparison of $f'(\eta)$ for $\lambda = 3/4$ for present methods and RungeKutta solution.

η	Runge-Kutta solution [55]	RGC	Error (RGC)	MGLF	Error (MGLF)
0	0.852193	0.852153	0.000040	0.852287	0.000094
0.1	0.755377	0.755336	0.000041	0.755347	0.000030
0.2	0.665448	0.665405	0.000043	0.665243	0.000205
0.3	0.582985	0.582891	0.000094	0.582644	0.000341
0.4	0.508141	0.508021	0.000120	0.507787	0.000354
0.5	0.440849	0.440760	0.000089	0.440604	0.000245
0.6	0.380907	0.380868	0.000039	0.380812	0.000095
0.7	0.327973	0.327951	0.000022	0.327984	0.000011
0.8	0.281536	0.281513	0.000023	0.281607	0.000071
0.9	0.241013	0.241001	0.000012	0.241122	0.000109
1	0.205832	0.205838	0.000006	0.205957	0.000125
1.1	0.175434	0.175452	0.000018	0.175549	0.000115
1.2	0.149275	0.149293	0.000018	0.149354	0.000079
1.3	0.126821	0.126846	0.000025	0.126867	0.000046
1.4	0.107596	0.107639	0.000043	0.107619	0.000023
1.5	0.091196	0.091241	0.000045	0.091186	0.000010

Graphs of the approximations of $f'(\eta)$ for different values of λ by rational Gegenbauer collocation method and modified generalized Laguerre collocation method are shown in Figs. 2 and 3, respectively. In this case we avoided comparing our solutions with the numerical solution because of having a good agreement.

The logarithmic graphs of the $\|Res\|^2$ at $\lambda = 1/2$ for rational Gegenbauer and modified generalized Laguerre collocation methods are shown in Fig. 4. This graph illustrates the convergence rate of the methods. Furthermore, it shows that the accuracy level of the RGC method is higher in this problem.

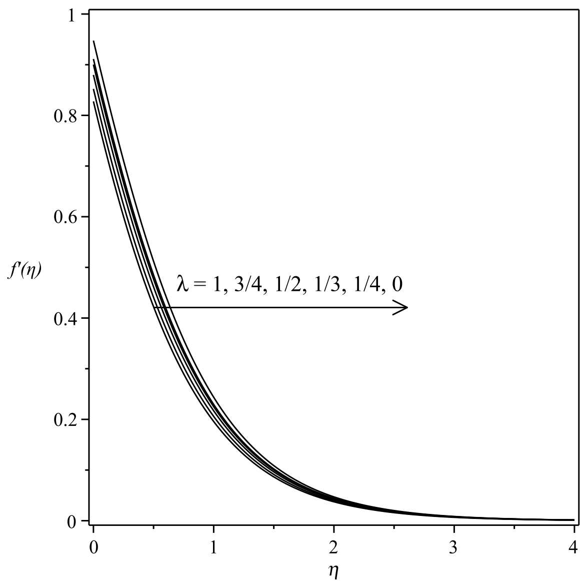

Fig. 2. RGC approximation of $f'(\eta)$ for different values $\lambda = 0, 1/4, 1/3, 1/2, 3/4$ and 1.

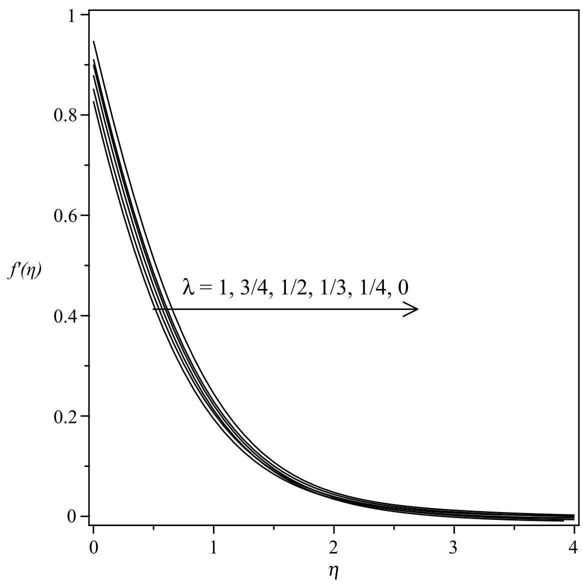

Fig. 3. MGLF approximation of $f'(\eta)$ for different values $\lambda = 0, 1/4, 1/3, 1/2, 3/4$ and 1.

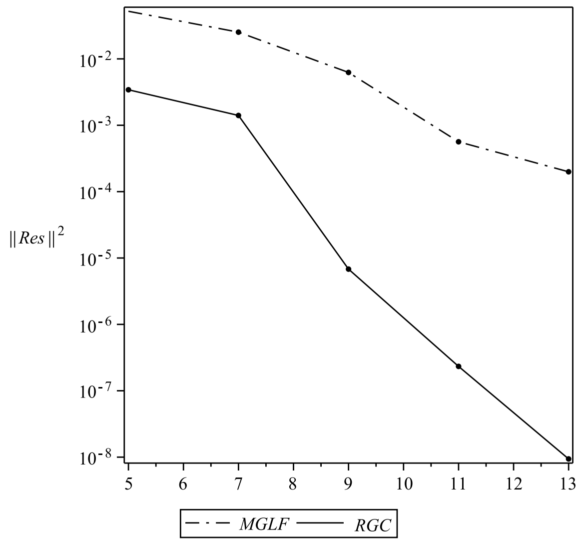

Fig. 4. Graph of $\|Res\|^2$ by RGCs solution and MGLFs solution.

5 Conclusions

In the above discussion, we applied the collocation method to solve three order nonlinear differential equations arising from the similarity solution of inverted cone, embedded in porous medium. This method is easy to implement and yields the desired accuracy. An important concern of collocation the approach is the choice of basis functions. The basis functions have three different properties: easy computation, rapid convergence and completeness, which means that any solution can be presented to arbitrarily high accuracy by taking the truncation N to be sufficiently large. We used two set of orthogonal functions as the basis function in this method and compared the results. Through the comparisons between the Runge-Kutta solutions [55] and the current works, it was shown that the present works provided acceptable approach for this type of equations. Although both functions lead to more accurate results, it seems that the accuracy and rapidity of RGC method is higher than those of MGLF method in this problem.

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